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RELATIONSHIP BETWEEN THE MILNOR'S μ-INVARIANT AND HOMFLYPT POLYNOMIAL

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1. INTRODUCTION

For an ordered oriented link in the 3-sphere, J. Milnor [15, 16] defined a family of invariants, known as *Milnor's $\overline{\mu}$-invariants.* For an n-component link L, Milnor invariant is determined by a sequence I of elements in $\{1, 2, \ldots, n\}$ and denoted by $\overline{\mu}_L(I)$. It is known that Milnor invariants of length two are just linking numbers. In general, Milnor invariant $\overline{\mu}_L(I)$ is only well-defined modulo the greatest common divisor $\Delta_L(I)$ of all Milnor invariants $\overline{\mu}_L(J)$ such that J is a subsequence of I obtained by removing at least one index or its cyclic permutation. If the sequence is of distinct numbers, then this invariant is also a link-homotopy invariant and we call it *Milnor's link-homotopy invariant.* Here, the *link-homotopy* is an equivalence relation generated by ambient isotopy and self-crossing changes.

In [3], N. Habegger and X. S. Lin showed that Milnor invariants are also invariants for string links, and these invariants are called Milnor's μ-invariants. For any string link σ, $\mu_\sigma(I)$ coincides with $\bar{\mu}_{\hat{\sigma}}(I)$ modulo $\Delta_{\hat{\sigma}}(I)$, where $\hat{\sigma}$ is a link obtained by the closure of σ. Milnor's μ-invariants of length k are finite type invariants of degree $k-1$ for any natural integer k, as shown by D. Bar-Natan [1] and X. S. Lin [11].

In [17], M. Polyak gave a formula expressing Milnor's $\bar{\mu}$-invariant of length 3 by the Conway polynomials of knots. His idea was derived from the following relation. Both Milnor's μ-invariant of length 3 for string link and the second coefficient of the Conway polynomial are finite type invariants of degree 2. He gave this relation by using Gauss diagram formulas.

Then, in [14], J-B. Meilhan and A. Yasuhara generalized it by using the clasper theory introduced by K. Habiro [4]. They showed that general Milnor's $\bar{\mu}$-invariants can be represented by the HOMFLYPT polynomials of knots under some assumption. Moreover the author and A. Yasuhara improved it in [9].

In [8], we give a formula expressing Milnor's μ-invariant by the HOMFLYPT polynomials of knots under some assumption (Theorem 3.1) by using the clasper theory in [4]. The course of proof is similar to that in [14] and [9]. Moreover, Milnor's μ-invariants of length 3 for any string link are given by the HOMFLYPT polynomial, which is a finite type invariant of degree 2, and the linking number. Because a finite type knot invariant of degree 2 is only the second coefficient of the Conway polynomial essentially, Milnor's μ-invariants of length 3 are given by the second coefficient of the Conway polynomial and the linking number (Theorem 3.3). It is a string version of Polyak's result, and by taking modulo $\triangle(I)$, our result coincides with Polyak's result.

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2. Milnor's μ-invariant and HOMFLYPT polynomial

2.1. **String link.** Let n be a positive integer and $D^2 \subset \mathbb{R}^2$ the unit disk equipped with n marked points $x_1, x_2, \ldots, x_n$ in its interior, lying in the diameter on the x-axis of $\mathbb{R}^2$ as in Figure 1. Let $I = [0,1]$. An *n-string link* σ is the image of a proper embedding $\sqcup_{i=1}^n I_i \to D^2 \times I$ of the disjoint union of n copies of I in $D^2 \times I$, such that $\sigma|_{I_i}(0) = (x_i, 0)$ and $\sigma|_{I_i}(1) = (x_i, 1)$ for each i as in Figure 1. Each string of a string link inherits an orientation from the usual orientation of I. The n-string link $\{x_1, x_2, \ldots, x_n\} \times I$ in $D^2 \times I$ is called the *trivial n-string link* and denoted by $\mathbf{1}_n$ or $\mathbf{1}$ simply.

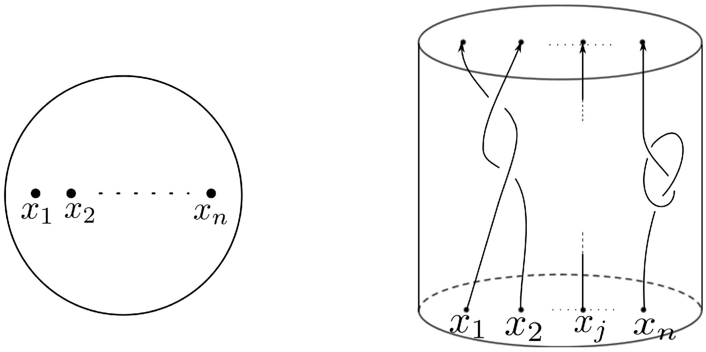


FIGURE 1. An n-string link

Given two n-string links σ and σ', we denote their product by $\sigma \cdot \sigma'$, which is given by stacking σ' on the top of σ and reparametrizing the ambient cylinder $D^2 \times I$. By this product, the set of isotopy classes of n-string links has a monoid structure with unit given by the trivial string link $\mathbf{1}_n$. Moreover, the set of link-homotopy classes of n-string links is a group under this product.

2.2. **Milnor's μ-invariant for string links.** Let $\sigma = \cup_{i=1}^n \sigma_i$ in $D^2 \times I$ be an n-string link. We consider the fundamental group $\pi_1(D^2 \times I \setminus \sigma)$ of the complement of σ in $D^2 \times I$, where we choose a point b as a base point and curves $\alpha_1, \cdots, \alpha_n$ as meridians in Figure 2.

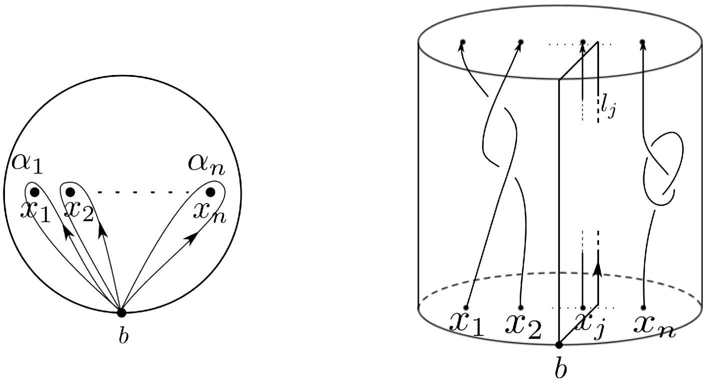


FIGURE 2. Longitude of string link

By Stallings' theorem [18], for any positive integer q, the inclusion map

$$\iota : D^2 \times \{0\} \setminus \{x_1, \cdots, x_n\} \longrightarrow D^2 \times I \setminus \sigma$$

induce an isomorphism of the lower central series quotients of the fundamental groups

$$\iota_* : \frac{\pi_1(D^2 \times \{0\} \setminus \{x_1, \cdots, x_n\})}{(\pi_1(D^2 \times \{0\} \setminus \{x_1, \cdots, x_n\}))_q} \longrightarrow \frac{\pi_1(D^2 \times I \setminus \sigma)}{\pi_1(D^2 \times I \setminus \sigma)_q},$$

where given a group G, G_q means the q-th lower central subgroup of G. The fundamental group $\pi_1(D^2 \times \{0\} \setminus \{x_1, \cdots, x_n\})$ is a free group generated by $\alpha_1, \cdots, \alpha_n$. We then consider the j-th longitude l_j of σ in $D^2 \times I$, where l_j is the closure of the preferred parallel curve of σ_j, whose endpoints lie on the x-axis in $D^2 \times \{0, 1\}$ as in Figure 2. We then consider the image of the longitude $\iota_*^{-1}(l_j)$ by the Magnus expansion and denote $\mu(i_1, \cdots, i_k, j)$ the coefficient of $X_{i_1} X_{i_2} \cdots X_{i_k}$ in the Magnus expansion.

Theorem 2.1 ([3]). *For any positive integer q, if $k < q$, then $\mu(i_1, \cdots, i_k, j)$ is invariant under isotopy. Moreover, if the sequence $i_1, \cdots, i_k, j$ is of distinct numbers, then $\mu(i_1, \cdots, i_k, j)$ is also link-homotopy invariant.*

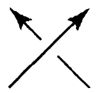
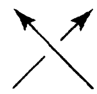
We call this invariant Milnor's μ-invariant.

2.3. **HOMFLYPT polynomial.** Recall the definition of the HOMFLYPT polynomial.

The *HOMFLYPT polynomial* $P(L; t, z) \in \mathbb{Z}[t^{\pm 1}, z^{\pm 1}]$ of an oriented link L is defined by the following two formulas:

(1) $P(U; t, z) = 1$, and
(2) $t^{-1}P(L_+; t, z) - tP(L_-; t, z) = zP(L_0; t, z)$,

where U denotes the trivial knot and L_+, L_- and L_0 are link diagrams which are identical everywhere except near one crossing, where they look as follows:

$L_+ =$  ; $L_- =$  ; $L_0 =$  .

Recall that the HOMFLYPT polynomial of a knot K is of the form $P(K; t, z) = \sum_{k=0}^{N} P_{2k}(K; t) z^{2k}$, where $P_{2k}(K; t) \in \mathbb{Z}[t^{\pm 1}]$ is called the $2k$-th coefficient polynomial of K.

3. MAIN THEOREM

Given a sequence I of elements of $\{1, 2, \ldots, n\}$, $J < I$ will be used for any subsequence J of I, possibly I itself, and $|J|$ will denote the length of the sequence J.

Let σ be an n-string link. Given a sequence $I = i_1 i_2 \cdots i_m$ obtained from $12 \cdots n$ by deleting some elements, and a subsequence $J = j_1 j_2 \cdots j_k$ of I, we define a knot $\overline{\sigma_{I,J}}$ as the closure of the product $b_I \cdot \sigma_J$. Here σ_J is the m-string link obtained from σ by deleting the i-th string, for all $i \in \{1, 2, \cdots, n\} \setminus \{i_1, i_2, \cdots, i_m\}$ and replacing the i-th string with a trivial string underpassing all other components, for all $i \in \{i_1, i_2, \cdots, i_m\} \setminus \{j_1, j_2, \cdots, j_k\}$, and b_I is the m-braid associated with the permutation $b = \begin{pmatrix} i_1 & i_2 & \cdots & i_{m-1} & i_m \\ i_2 & i_3 & \cdots & i_m & i_1 \end{pmatrix}$ and such that the arc with connecting $(b^k(i_1), 0)$ with $(b^{k+1}(i_1), 1)$ underpasses all arcs with connecting $(b^{k'}(i_1), 0)$ with $(b^{k'+1}(i_1), 1)$ in $[0, 1] \times [0, 1]$ of braid diagram for $k < k' < n$. See Figure 3 for an example. We then have the following Theorem.

Theorem 3.1. *Let σ be an n-string link ($n \geq 4$) with vanishing Milnor's link-homotopy invariants of length $\leq m-2$. Then for any sequence I obtained from $12\cdots n$ by deleting $n-m$ elements, we have*

$$\mu_\sigma(I) = \frac{(-1)^{m-1}}{(m-1)!2^{m-1}} \sum_{J<I} (-1)^{|J|} P_0^{(m-1)}(\overline{\sigma_{I,J}}; 1),$$

where $P_0^{(m-1)}(\cdot; 1)$ is the $(m-1)$-th derivative of the 0-th coefficient $P_0(\cdot; t)$ of the HOMFLYPT polynomial evaluated at $t = 1$.

Note that the above vanishing assumption for string link is equivalent to that any $(m-2)$-substring link is link-homotopic to the trivial string link.

Remark 3.2. Theorem 1.1 remains valid if we use one of the following two alternative definitions of b_I. One is that we use "overpasses" instead of "underpasses". The other is that we use "any $i \in \{i_1, i_2, \cdots, i_m\}$" instead of "$i_1$".

We also give the case of μ-invariants of length 3 without the assumption.

Theorem 3.3. *Let σ be an n-string link and $I = i_1 i_2 i_3$ be a length 3 sequence with distinct numbers in $\{1, 2, \cdots, n\}$. We then have*

$$\mu_\sigma(I) = -\sum_{J<I} (-1)^{|J|} a_2(\overline{\sigma_{I,J}}) - lk_\sigma(i_1 i_2) lk_\sigma(i_2 i_3) + A_I,$$

where a_2 is the second coefficient of the Conway polynomial, $lk_\sigma(ij)$ is the linking number of the i-th component and j-th component of σ, and

$$A_I = \begin{cases} lk_\sigma(i_1 i_2) & (i_2 < i_3 < i_1) \\ -lk_\sigma(i_1 i_2) & (i_1 < i_3 < i_2) \\ 0 & (otherwise). \end{cases}$$

Remark 3.4. This operation from a string link to a knot corresponds to Y-graph sum of links defined by M. Polyak. By taking this formula modulo $\Delta_{\overline{\sigma_{I,J}}}(I)$, we get Polyak's relation between Milnor's $\overline{\mu}$-invariants and Conway polynomials [17].

Remark 3.5. In [19], K. Taniyama gave a formula expressing Milnor's $\overline{\mu}$-invariants of length 3 for links by the second coefficient of the Conway polynomial assuming that all linking numbers vanish.

Remark 3.6. In [12], J.B. Meilhan showed that all finite type invariants of degree 2 for string link was given a formula by some invariants (Theorem 2.8). So the formula in Theorem 3.3 could also be derived from [12].

4. Examples

Example 4.1. Let σ be a 3-string link showed by Figure 3. Then $\mu_{123}(\sigma) = -1$, $\mu_{132}(\sigma) = \mu_{213}(\sigma) = 1$ and $\mu_{231}(\sigma) = \mu_{312}(\sigma) = \mu_{321}(\sigma) = 0$. And $lk_\sigma(12) = lk_\sigma(23) = 1$ and $lk_\sigma(13) = 0$.

On the other hand, $\overline{\sigma_{123,123}}$ and $\overline{\sigma_{123,23}}$ are the figure-eight knot, and $\overline{\sigma_{123,J}}$ ($J \neq 123, 23$) is the trivial knot. Therefore we obtain

$$-\sum_{J<123} (-1)^{|J|} a_2(\overline{\sigma_{123,J}}) - lk_\sigma(12) lk_\sigma(23) = a_2(4_1) - a_2(4_1) - 1 \cdot 1 = -1.$$

Similarly, we have

$$-\sum_{J<231}(-1)^{|J|}a_2(\overline{\sigma_{231,J}})-lk_\sigma(23)lk_\sigma(31)=a_2(3_1\sharp 4_1)-a_2(3_1)-a_2(4_1)-1\cdot 0=0,$$

$$-\sum_{J<312}(-1)^{|J|}a_2(\overline{\sigma_{312,J}})-lk_\sigma(31)lk_\sigma(12)+lk_\sigma(13)=a_2(3_1)-a_2(3_1)-0\cdot 1+0=0.$$

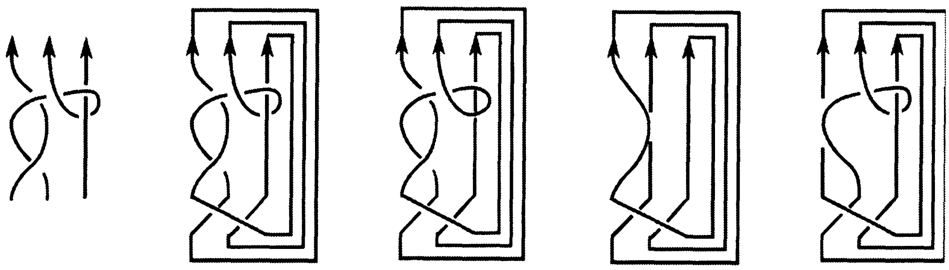
Moreover, $\overline{\sigma_{132,32}}$ is the figure-eight knot and $\overline{\sigma_{132,J}}$ $(J\neq 32)$ is the trivial knot. Therefore we obtain

$$-\sum_{J<132}(-1)^{|J|}a_2(\overline{\sigma_{132,J}})-lk_\sigma(13)lk_\sigma(32)-lk_\sigma(13)=-a_2(4_1)-0\cdot 1-0=1.$$

Similarly, we have

$$-\sum_{J<213}(-1)^{|J|}a_2(\overline{\sigma_{213,J}})-lk_\sigma(21)lk_\sigma(13)=a_2(7_6)-a_2(3_1)-a_2(4_1)-1\cdot 0=1,$$

$$-\sum_{J<321}(-1)^{|J|}a_2(\overline{\sigma_{321,J}})-lk_\sigma(32)lk_\sigma(21)=a_2(5_2)-a_2(3_1)-1\cdot 1=0.$$



σ $\quad\overline{\sigma_{123,123}}$ $\quad\overline{\sigma_{123,12}}$ $\quad\overline{\sigma_{123,13}}$ $\quad\overline{\sigma_{123,23}}$

FIGURE 3

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